WEAK SOLUTIONS FOR NONLOCAL PARABOLIC PROBLEMS OF KIRCHHOFF TYPE VIA TOPOLOGICAL DEGREE THEORY

Pham Van Du
Dong Nai University
Email: dupv@dnpu.edu.vn

(Ngày nhận bài: 13/6/2025, ngày nhận bài chỉnh sửa: 11/8/2025, ngày duyệt đăng: 24/10/2025)

ABSTRACT

We consider a class of nonlinear, nonlocal parabolic equations featuring a Kirchhoff-type term and variable exponent growth. These equations naturally arise in the modeling of complex physical processes such as electrorheological fluids and anisotropic diffusion, where the interplay between nonstandard growth conditions and nonlocal effects is particularly significant. Motivated by the recent contribution of M. Ait Hammou [Kragujevac J. Math, 47(4), 523–529 (2023)], who employed topological degree theory to address quasilinear parabolic problems involving the $p(x)$-Laplacian, we broaden this analytical framework to encompass Leray–Lions type operators with gradient dependence, coupled with Kirchhoff-type nonlocal coefficients. Building on this foundation, we employ a carefully refined version of the Berkovits–Mustonen degree theory to establish the existence of weak solutions in variable exponent Sobolev spaces. Our analysis relies on verifying key structural properties of the associated operators—such as continuity, boundedness, and the (S_+)-condition—which facilitate the application of a fixed point argument to operator sums. This work expands the applicability of topological methods to a broader class of nonlinear parabolic equations, including a wide range of models with nonstandard diffusion and nonlocal effects.

Keywords: *Nonlocal parabolic problems, $p(x)$ −Laplacian, topological degree, variable exponents*

1. Introduction

Nonlinear parabolic problems with variable exponent growth have gained significant attention in recent decades due to their capacity to model a wide range of physical phenomena, such as electrorheological fluids, non-Newtonian flows, and anisotropic diffusion processes [1], [2], [3]. A key class of such models involves the $p(x)$-Laplacian operator:

$$\Delta_{p(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right),$$

which has been extensively studied in the reflexive Banach space

$$V := L^{p(\cdot)}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right),$$

accommodating spatial variability in the growth conditions. Existence results in this framework are typically derived using monotonicity or variational techniques [4], [5].

A notable advance was made by Ait Hammou [6], who applied the topological degree theory of Berkovits and Mustonen [7] to establish the existence of weak solutions for a quasilinear parabolic problem involving the $p(x)$-Laplacian:

$$\begin{cases} u_t - \Delta_{p(x)}u = f & \text{in } Q := \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where the right-hand side f is understood as an element of the dual space V^*, with

$$V = \{u \in W^- \mid |\nabla u| \in L^{p(\cdot)}(Q)\}$$

, and $u_0 \in L^2(\Omega)$.

where $W^- := L^{p^-}\left(0,T; W_0^{1,p(\cdot)}(\Omega)\right)$.

The novel aspect of this approach lies in treating the problem through operator sums of the form $L + S$, where L is a linear maximal monotone operator, and S is a bounded, demicontinuous, and (S_+)-type mapping. This enabled the establishment of solvability in V without relying on variational structures.

At the conclusion of [6], a natural question was raised:

Can the degree-theoretic approach be extended to more general nonlinearities, such as those of the Leray–Lions type depending on the gradient?

This paper aims to answer this question affirmatively. We investigate a class of doubly nonlinear parabolic problems involving a nonlocal Kirchhoff-type coefficient and a Leray–Lions type operator. Specifically, we consider the problem (1.1) given by

$$\begin{cases} u_t - M_A(t)\operatorname{div} a = h & \text{in } Q := \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

Where

- $M: [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing Kirchhoff function,
- $A(x,t,\xi)$ and $a(x,t,\xi)$ are Carathéodory functions satisfying appropriate growth and monotonicity conditions,
- $h \in V^*$ is a given source term,
- $u_0 \in L^2(\Omega)$ is the initial datum,
- $M_A(t) := M\left(\int_\Omega A\,(x,t,\nabla u)\,dx\right)$,

  and $\operatorname{div} a := \operatorname{div} a(x,t,\nabla u)$.

This model extends that of Ait Hammou in two fundamental directions. First, the differential operator is of Leray–Lions type, exhibiting nonlinear dependence on the gradient. Second, it incorporates a nonlocal Kirchhoff coefficient M that depends on the total energy $\int_\Omega A\,(x,t,\nabla u)\,dx$, introducing both strong nonlinearity and nonlocal effects into the system.

By suitably adapting the topological degree framework of Berkovits and Mustonen, we establish the existence of weak solutions under general structural conditions. Our result significantly expands the scope of degree-theoretic methods to encompass broader classes of nonlinear evolution equations, and demonstrates their effectiveness in handling physically relevant models involving gradient-dependent and nonlocal nonlinearities.

2. Literature Review

Nonlinear parabolic problems with variable exponent growth have attracted increasing attention in recent decades due to their ability to model electrorheological fluids, non-Newtonian flows, and anisotropic diffusion processes [1], [2], [3]. Central to this field is the $p(x)$-Laplacian operator, which has been studied extensively in variable-exponent Sobolev spaces, where existence results are typically derived using monotonicity and variational arguments [4], [5]. A significant breakthrough was achieved by Ait Hammou [6], who employed the topological degree theory of Berkovits and Mustonen [7] to establish weak solvability for quasilinear parabolic problems without relying on variational structures. However, this approach leaves open the extension to more general Leray–Lions type operators and, in particular, to Kirchhoff-type problems where the coefficients depend nonlocally on the total energy. This gap is especially relevant since renormalized solutions and existence results for variable exponent and nonlocal frameworks remain a topic of current investigation [8], [9], [10]. Motivated by this, the present work adapts the degree-theoretic framework to parabolic equations driven by a Leray–Lions operator coupled with a Kirchhoff-type coefficient, thereby expanding the applicability of topological methods to a broader class of nonlinear and nonlocal evolution problems.

3. Theoretical basis

3.1. Function Space Setting with Variable Exponents

We work within the framework of variable exponent Lebesgue and Sobolev spaces, which provide a natural setting for nonlinear problems with nonstandard growth. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded Lipschitz domain, and let $p: \overline{\Omega} \to [1, \infty)$ be a measurable function satisfying

$$1 < p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty.$$

The space $L^{p(\cdot)}(\Omega)$ consists of measurable functions u such that

$$\int_\Omega |u(x)|^{p(x)}\, dx < \infty,$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0: \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The corresponding Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ consists of functions in $L^{p(\cdot)}(\Omega)$ with weak gradient in $\left(L^{p(\cdot)}(\Omega)\right)^N$ and with zero trace on the boundary $\partial\Omega$, endowed with the norm $\|u\| := \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$. These spaces are reflexive, separable, and uniformly convex; their duals are $L^{p'(\cdot)}(\Omega)$ and $W^{-1,p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Under the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \text{ for } |x - y| \leq \frac{1}{2},$$

the space $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$, and a generalized Poincaré inequality holds.

To incorporate time dependence, we consider the space-time cylinder Q

$:= \Omega \times (0,T)$ for some $T > 0$, with the extended exponent $p(x,t) := p(x)$. Following [8], we define the evolution space

$$V := \{u \in W^-: |\nabla u| \in L^{p(\cdot)}(Q)\},$$

which provides a natural setting for studying nonlinear problems involving nonstandard growth. The space V is a reflexive and separable Banach space, equipped with the norm

$$\|u\|_V := \|u\|_{W^-} + \|\nabla u\|_{L^{p(\cdot)}(Q)}$$

or, alternatively, with the equivalent norm

$$\|u\|_V := \|\nabla u\|_{L^{p(\cdot)}(Q)}.$$

Moreover, the space V satisfies the continuous embeddings

$$W^+ \hookrightarrow V \hookrightarrow W^-,$$

where $W^+ := L^{p^+}\left(0,T; W_0^{1,p(\cdot)}(\Omega)\right)$.

In turn, its dual space V^* satisfies

$$W_{-1}^+ \hookleftarrow V^* \hookleftarrow W_{-1}^-,$$

where $W_{-1}^+ := L^{(p^+)'}\left(0,T; W^{-1,p'(\cdot)}(\Omega)\right)$ and $W_{-1}^- := L^{(p^-)'}\left(0,T; W^{-1,p'(\cdot)}(\Omega)\right)$.

3.2. Classes of Multivalued Mappings and the Topological Degree

Let X be a real, separable, reflexive Banach space with topological dual X^*, and let $\langle\cdot,\cdot\rangle$ denote the canonical duality pairing. We denote by $u_n \rightharpoonup u$ the weak convergence in X.

We consider multivalued mappings $T: X \to 2^{X^*}$, associating with each such mapping its graph:

$$G(T) := \{(u,w) \in X \times X^*: w \in T(u)\}.$$

The mapping T is said to be monotone if for every $(u_1,w_1),(u_2,w_2) \in G(T)$, one has

$$\langle w_1 - w_2, u_1 - u_2\rangle \geq 0.$$

It is said to be maximal monotone if its graph is maximal with respect to inclusion among all monotone mappings. That is, if $(u_0,w_0) \in X \times X^*$ satisfies

$$\langle w_0 - w, u_0 - u\rangle \geq 0, \forall (u,w) \in G(T),$$

then necessarily $(u_0,w_0) \in G(T)$.

Let $T: D(T) \subset X \to Y$ be a mapping. It is called demicontinuous if $u_n \to u$ strongly in X implies $T(u_n) \rightharpoonup T(u)$ weakly in Y. The mapping T is said to be of class (S_+) if

$$u_n \rightharpoonup u \text{ in } D(T) \text{ and}$$
$$\limsup_{n\to\infty}\langle T(u_n), u_n - u\rangle \leq 0$$

imply that $u_n \to u$.

Let $L: D(L) \subset X \to X^*$ be a linear maximal monotone mapping with dense domain $D(L)$. Let $G \subset X$ be a nonempty open bounded subset. Define the following operator classes:

The class $\mathcal{F}_G$ consists of all operators of the form

$$F = L + S: \overline{G} \cap D(L) \to X^*,$$

where S is bounded, demicontinuous, and of class (S_+).

The class $\mathcal{H}_G$ consists of all homotopies of the form

$$F(t) = L + S(t): \overline{G} \cap D(L) \to X^*, t \in [0,1],$$

where $S(t)$ is a bounded family of demicontinuous operators of class (S_+) depending continuously on t.

It is worth noting that $\mathcal{H}_G$ includes all affine homotopies of the form

$$F(t) = L + (1-t)S_1 + tS_2,$$

where $L + S_i \in \mathcal{F}_G,\ i = 1,2$.

The following theorem, due to Berkovich and Mustonen, establishes a well-defined notion of topological degree for the operator class $\mathcal{F}_G$, extending classical constructions to monotone-type operators of the form $L + S$. This generalized framework provides a robust tool for analyzing solvability in nonlinear problems.

Theorem 3.1 (Topological Degree for $\mathcal{F}_G$)

Let L be a linear maximal monotone operator defined on a dense domain $D(L) \subset X$. Then there exists a topological degree function

$$d: U \to \mathbb{Z}$$

where

$$U := \{(F, G, h): F \in \mathcal{F}_G,\ G \subset X \text{ open and bounded},\ h \notin F(\partial G \cap D(L))\}$$

satisfying the following properties:

(a) ***Existence:*** *If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has at least one solution in $G \cap D(L)$.*

(b) ***Additivity:*** *If $G_1, G_2 \subset G$ are disjoint open sets such that*

$$h \notin F\left(\overline{G_1 \cup G_2} \setminus (G_1 \cup G_2) \cap D(L)\right),$$

then

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

(c) ***Homotopy invariance:*** *If $F(t) \in \mathcal{H}_G$ and $h(t) \notin F(t)(\partial G \cap D(L))$ for all $t \in [0,1]$, where $h(t): [0,1] \to X^*$ is a continuous curve, then the degree is constant along the homotopy:*

$$d(F(t), G, h(t)) = const \text{ for all } t \in [0,1].$$

(d) ***Normalization:*** *Let J be the duality mapping of X into X^*. Then*

$$d(L + J, G, h) = 1, \quad \text{whenever } h \notin (L + J)(\partial G \cap D(L)).$$

A principal application of this theory is the derivation of surjectivity criteria for operators in $\mathcal{F}_X$, that is, conditions ensuring that the range of $F = L + S$ covers the entire dual space X^*. Theorem 3.2 provides a first such criterion, establishing full-range properties under a coercivity-type condition on the boundary of a ball in X [6].

Theorem 3.2 (Surjectivity Criterion)

Let $F = L + S \in \mathcal{F}_X$ and let $h \in X^*$. Suppose there exists $R > 0$ such that

$$\langle Lu + Su - h, u\rangle > 0 \quad \text{for all } u \in \partial B_R(0) \cap D(L).$$

Then the image of F is the whole space:

$$(L + S)(D(L)) = X^*.$$

In many applications, however, such pointwise coercivity conditions on the boundary may be difficult to verify or overly restrictive. To address this, Theorem 3.3 offers a more flexible criterion, replacing the boundary condition with a growth assumption at infinity. This formulation is particularly well-suited to nonlinear partial differential equations and variational problems, where coercivity at large norms often arises naturally from energy estimates.

Theorem 3.3. *Let $L + S \in \mathcal{F}_X$, and $h \in X^*$. Suppose that there exist $R > 0$ and a* *continuous function* $\phi: [0,\infty) \to \mathbb{R}$ *such that:*

$$\langle Lu + Su - h, u\rangle \geq \phi(\|u\|), \forall u \in D(L) \text{ with } \|u\| \geq R,$$

and $\phi(r) > 0$ *for all* $r \geq R$. *Then,*

$$(L+S)\big(D(L)\big) = X^*.$$

Proof. Let $\varepsilon > 0$, $t \in [0,1]$, and consider the family of perturbed operators

$$F_\varepsilon(t,u) := Lu + (1-t)Ju + t(Su + \varepsilon Ju - h).$$

Since J is the duality mapping on X, we have $\langle Ju, u\rangle = \|u\|^2$. Note also that $0 \in L(0)$, and that both J and $S + \varepsilon J$ are continuous, bounded and of class (S_+). Thus, the family $\{F_\varepsilon(t,\cdot)\}_{t\in[0,1]}$ defines an admissible homotopy.

Let $u \in \partial B_R(0) \cap D(L)$. By assumption, we have

$$\langle Lu + Su - h, u\rangle \geq \phi(\|u\|) = \phi(R) > 0.$$

It follows that

$$\langle F_\varepsilon(t,u), u\rangle = \langle t(Lu + Su - h), u\rangle + \langle (1-t)Lu + (1 - t + t\varepsilon)Ju, u\rangle \geq t\cdot\phi(R) + (1-t)\langle Lu, u\rangle + (1 - t + t\varepsilon)\langle Ju, u\rangle.$$

As $\langle Ju, u\rangle = \|u\|^2 = R^2$, we conclude

$$\langle F_\varepsilon(t,u), u\rangle \geq t\cdot\phi(R) + (1 - t + t\varepsilon)R^2 > 0.$$

Therefore, $0 \notin F_\varepsilon\big(t, \partial B_R(0)\big)$, and the degree

$$d(F_\varepsilon(t,\cdot), B_R(0), 0)$$

is well-defined and constant for all $t \in [0,1]$. In particular, by the normalization property of the degree, we obtain

$$d(F_\varepsilon(t,\cdot), B_R(0), 0) = d(L + J, B_R(0), 0) = 1.$$

Hence, for each $\varepsilon > 0$, there exists $u_\varepsilon \in D(L)$ such that

$$0 \in F_\varepsilon(1, u_\varepsilon) = Lu_\varepsilon + Su_\varepsilon + \varepsilon Ju_\varepsilon - h.$$

That is,

$$Lu_\varepsilon + Su_\varepsilon = h - \varepsilon Ju_\varepsilon.$$

Since $\varepsilon \to 0^+$, and $\|Ju_\varepsilon\| = \|u_\varepsilon\|$, we may extract a subsequence (still denoted u_ε) such that Ju_ε remains bounded in X^*. Therefore, taking $\varepsilon \to 0^+$, we conclude the existence of $u \in D(L)$ such that

$$Lu + Su = h.$$

As $h \in X^*$ is arbitrary, we conclude that

$$(L+S)\big(D(L)\big) = X^*.$$

This completes the proof. Q.E.D.

4. Basic Assumptions and Main Result

We consider a nonlinear operator of the form

$$a(x,t,\xi): Q \times \mathbb{R}^N \to \mathbb{R}^N,$$

which is assumed to be a Carathéodory mapping and satisfies the following structural conditions:

(h1) (Growth condition):

$$|a(x,t,\xi)| \leq k_1(x,t) + c_1|\xi|^{p(x)-1},$$

for all $(x,t) \in Q$, $\xi \in \mathbb{R}^N$, where $k_1 \in L^{p'(\cdot)}(Q)$, $c_1 > 0$, and $p(x) > 1$.

(h2) (Strict monotonicity):

$$\big(a(x,t,\xi) - a(x,t,\xi')\big)\cdot(\xi - \xi') > 0,$$

for all $(x,t) \in Q$, and all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(h3) (Coercivity):

$$a(x,t,\xi)\cdot\xi \geq c_2|\xi|^{p(x)} - k_2(x,t) > 0,$$

for all $(x,t) \in Q$, $\xi \in \mathbb{R}^N$, with $k_2 \in L^1(Q)$ and $c_2 > 0$.

In addition, we assume that $a(x,t,\xi)$ is the gradient with respect to ξ of a continuous potential mapping $A(x,t,\xi): Q \times \mathbb{R}^N \to \mathbb{R}^N$, i.e.,

(h4) $A(x,t,0) = 0$ and $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$;

(h5) $a(x,t,\xi) \cdot \xi \le p(x) A(x,t,\xi)$, for all $(x,t) \in Q, \xi \in \mathbb{R}^N$.

To address the existence of weak solutions, we introduce a Kirchhoff-type function

$$M: \mathbb{R}^+ \to \mathbb{R}^+,$$

which is assumed to be continuous and non-decreasing. We also assume that the functions $u(\cdot,t), v(\cdot,t) \in W^{1,p(x)}(Q)$ are such that the following condition holds:

(M_0) There exist positive constants m_0, m_1 such that

$$m_0 \le M(t) \le m_1, \quad \text{for all } t \in [0,+\infty).$$

Under the structural assumptions (h1)–(h3) on the nonlinear operator $a(x,t,\xi)$, notably its coercivity and quasi-strict monotonicity, we can establish a strong convergence result for sequences that converge weakly. This property is fundamental in proving the stability and continuity of solutions with respect to variations in the data. The lemma below formalizes this key result. Its proof follows similar lines to the constant exponent setting; for further details, refer to [9] and [10]

Lemma 4.1

Suppose that assumptions (h1)–(h3) hold. Let $(u_n) \subset V$ be a sequence such that $u_n \rightharpoonup u$ weakly in V, and suppose that

$$\int_Q \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big) \cdot (\nabla u_n - \nabla u)\, dx\, dt \to 0.$$

Then $u_n \to u$ strongly in V.

The following lemma is crucial to the proof of the main result of the paper, as it ensures that the nonlinear operator S is bounded, continuous, and of class (S_+). These properties are key to applying monotonicity-based existence results, allowing us to handle the nonlocal structure of the equation and establish the existence of weak solutions.

Lemma 4.2

Define the operator $S: V \to V^*$ by

$$\langle Su, v\rangle := \int_0^T M_A(t) \int_\Omega a(x,t,\nabla u) \times \nabla v\, dx\, dt,$$

$\forall u, v \in V$.

Under assumptions (M_0) and (h1)–(h5), the operator S enjoys the following properties:

(i) *The operator S is bounded and continuous from V into V^*.*

(ii) *The operator S is of class (S_+).*

Proof. We divide the proof into two parts corresponding to the two claims.

Part (i): Boundedness and Continuity of S

Let $u \in V$ be arbitrary. From the definition of S, for any $v \in V$, we have $|\langle Su, v\rangle| \le \int_0^T M_A(t) \int_\Omega |a(x,t,\nabla u)| \times |\nabla v|\, dx\, dt.$

Using the growth condition (h1), we estimate

$|a(x,t,\nabla u)| \le k_1(x,t) + c_1 |\nabla u|^{p(x)-1},$

where $k_1 \in L^{p'(\cdot)}(Q)$. Then by Hölder's inequality in variable exponent spaces

$$\int_Q |a(x,t,\nabla u)| \cdot |\nabla v| \, dx \, dt \le \int_Q (k_1(x,t) + c_1) \, |\nabla v| \, dx \, dt \le C \left(\|k_1\|_{L^{p'(\cdot)}(Q)} + \left\| |\nabla u|^{p(x)-1} \right\|_{L^{p'(\cdot)}(Q)} \right) \times \|\nabla v\|_{L^{p(\cdot)}(Q)}$$

Now using assumption (M_0), namely $M(t) \le m_1$, we obtain

$$|\langle Su, v\rangle| \le m_1 C \left(\|k_1\|_{L^{p'(\cdot)}(Q)} + \max\left\{ \|\nabla u\|^{p^- -1}_{L^{p(\cdot)}(Q)} \, ; \, \|\nabla u\|^{p^+ -1}_{L^{p(\cdot)}(Q)} \right\} \right) \|\nabla v\|_{L^{p(\cdot)}(Q)}.$$

Therefore,

$$\|Su\|_{V^*} \le m_1 C \left(\|k_1\|_{L^{p'(\cdot)}(Q)} + \max\left\{ \|\nabla u\|^{p^- -1}_{L^{p(\cdot)}(Q)} \, ; \, \|\nabla u\|^{p^+ -1}_{L^{p(\cdot)}(Q)} \right\} \right),$$

which implies that S is bounded.

To prove continuity, we let $(u_n) \subset V$ be a sequence such that

$$u_n \to u \quad \text{strongly in } V.$$

According to the definition of the norm on V, this implies

$$\int_Q |\nabla u_n - \nabla u|^{p(x)} \, dxdt \to 0.$$

In particular, up to a subsequence, we have pointwise convergence

$$\nabla u_n(x,t) \to \nabla u(x,t) \quad \text{for a.e. } (x,t) \in Q,$$

and, by the uniform integrability implied by strong convergence, there exists a function $g \in L^1(Q)$ such that

$$|\nabla u_n - \nabla u|^{p(x)} \le g(x,t)$$

for all n and a.e. (x,t).

From the growth condition (h1) on a, there exists $C > 0$ such that

$$|a(x,t,\nabla u_n)|^{p'(x)} \le C\left(|k_1(x,t)|^{p'(x)} + |\nabla u_n|^{p(x)} \right),$$

where $k_1 \in L^{p'(\cdot)}(Q)$ is given.

Using the convexity of $\xi \mapsto |\xi|^{p(x)}$ and the triangle inequality in modular form, we have

$$|\nabla u_n|^{p(x)} \le 2^{p^+ -1}\left(|\nabla u|^{p(x)} + |\nabla u_n - \nabla u|^{p(x)} \right) \le 2^{p^+ -1}\left(|\nabla u|^{p(x)} + g \right).$$

Since $|\nabla u|^{p(x)} \in L^1(Q)$ and $g \in L^1(Q)$, the right side is integrable. Consequently, $(|a(x,t,\nabla u_n)|^{p'(x)})_{n\in\mathbb{N}}$ is dominated by an integrable function independent of n, hence it is bounded in $L^1(Q)$ and and uniformly integrable.

Because $a(x,t,\cdot)$ is continuous a.e. $(x,t) \in Q$, and

$$\nabla u_n(x,t) \to \nabla u(x,t) \quad \text{a.e. in } Q,$$

by Vitali's convergence theorem we obtain

$$a(x,t,\nabla u_n) \to a(x,t,\nabla u) \quad \text{strongly in } L^{p'(\cdot)}(Q)^N.$$

Define for each $t \in (0,T)$

$$A_n(x,t) := a\big(x,t,\nabla u_n(x,t)\big) \cdot \nabla u_n(x,t), \quad A(x,t) := a\big(x,t,\nabla u(x,t)\big) \cdot \nabla u(x,t).$$

By Hölder's inequality in variable exponent spaces and the above strong convergences, it follows that

$$A_n \to A \quad \text{strongly in } L^1(Q).$$

Hence, for a.e. $t \in (0,T)$,

$$\int_\Omega A_n\,(x,t)\,dx \to \int_\Omega A\,(x,t)\,dx.$$

Since $M\colon \mathbb{R} \to \mathbb{R}$ is continuous and bounded on bounded sets, define

$$M_n(t) := M\left(\int_\Omega A_n\,(x,t)\,dx\right).$$

Then,

$$M_n(t) \to M_A(t) \quad \text{for a.e. } t \in (0,T) \text{ and } |M_n(t)| \le m_1,$$

for some constant $m_1 > 0$.

Let $v \in V$ with $\|v\|_V \le 1$. We write $|\langle S(u_n) - S(u), v\rangle|$
$= \left|\int_0^T [M_n(t) \int_\Omega a\,(x,t,\nabla u_n) \cdot \nabla v\,dx) - M_A(t) \int_\Omega a\,(x,t,\nabla u) \cdot \nabla v\,dx]dt\right|$
$\le \int_0^T |M_n(t)| \int_\Omega |a(x,t,\nabla u_n) - a(x,t,\nabla u)|\,.\,|\nabla v|\,dxdt + \int_0^T |M_n(t) - M_A(t)| \int_\Omega |a(x,t,\nabla u)|\,.\,|\nabla v|\,dxdt$
$= I_1 + I_2.$

By Hölder's inequality and the boundedness of $\|v\|_V$,
$I_1 \le m_1 \|a(x,t,\nabla u_n) - a(x,t,\nabla u)\|_{L^{p'(\cdot)}(Q)} \times \|\nabla v\|_{L^{p(\cdot)}(Q)}$
$\le m_1 \|a(x,t,\nabla u_n) - a(x,t,\nabla u)\|_{L^{p'(\cdot)}(Q)} \to 0$

Since
$t \mapsto \int_\Omega |a(x,t,\nabla u)|\;|\nabla v|\,dx \in L^1(0,T)$,
and $M_n(t) \to M_A(t)$ a.e. with uniform boundedness, the dominated convergence theorem implies

$$I_2 \to 0.$$

Part (ii): The operator S is of class (S_+).

Let $u_n \rightharpoonup u$ weakly in V, and suppose

$$\limsup_{n\to\infty} \langle Su_n, u_n - u\rangle \le 0.$$

From the definition of S, we have
$\langle Su_n, u_n - u\rangle = \int_0^T M_n(t) \int_\Omega a\,(x,t,\nabla u_n)(\nabla u_n - \nabla u)\,dx\,dt.$

Defne the functions
$\phi_n(t) := \int_\Omega \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\,(\nabla u_n - \nabla u)\,dx.$

We estimate
$\langle Su_n, u_n - u\rangle =$
$\int_0^T M_n(t) \int_\Omega \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\,(\nabla u_n - \nabla u)\,dx\,dt$
$+ \int_0^T M_n(t) \int_\Omega a(x,t,\nabla u)\,(\nabla u_n - \nabla u)\,dx\,dt$
$= \int_0^T M_n(t)\phi_n(t)\,dt + \int_0^T M_n(t) \int_\Omega a(x,t,\nabla u)\,(\nabla u_n - \nabla u)\,dx\,dt.$

The second term tends to zero due to the weak convergence $\nabla u_n \rightharpoonup \nabla u$ in $L^{p(\cdot)}(Q)^N$ and the growth bound (h1), which ensures $a(x,t,\nabla u) \in L^{p'(\cdot)}(Q)^N$, while the uniform boundedness of $M_n(t)$ guaranteed by (M_0) preserves this convergence. Consequently
$\limsup_{n\to\infty} \int_0^T M_n(t) \int_\Omega \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\,(\nabla u_n - \nabla u)\,dx\,dt \le 0.$

Applying the assumption (h2) with $\xi = \nabla u_n(x,t)$ and $\eta = \nabla u(x,t)$, we see that the integrand is nonnegative and strictly positive on any subset of positive measure where $\nabla u_n \ne \nabla u$. Thus,
$m_0 \int_0^T \varphi_n\,(t)\,dt \le \int_0^T M_n(t)\,\varphi_n(t)\,dt,$
and this implies
$\lim_{n\to\infty} \int_0^T \varphi_n\,(t)\,dt = 0.$

Now, invoking Lemma 4.1, we deduce that $u_n \to u$ strongly in V. Hence, S is of class (S_+). Q.E.D.

Theorem 4.3 (Existence of Weak Solutions)

Let $h \in V^*$ and $u_0 \in L^2(\Omega)$. Then there exists at least one weak solution $u \in D(L)$ to problem (1.1), in the sense that for all $v \in V$, the following identity holds

$$-\int_Q u\, v_t\, dxdt + \int_0^T M_A(t) \int_\Omega a(x,t,\nabla u)\nabla v\, dx\, dt = \int_Q h\, v\, dxdt.$$

Proof.

We first define the linear operator $L: V \supset D(L) \to V^*$, corresponding to the time derivative in distributional sense, by

$$D(L) := \{v \in V: v' \in V^*,\ v(0) = 0\}, \text{ and}$$

$$\langle Lu, v\rangle := -\int_Q u\, v_t\, dxdt = \int_0^T \langle u(t), v(t)\rangle\, dt.$$

for all $u \in D(L)$ and $v \in V$.

It is known that L is a densely defined, maximal monotone operator in V, as shown in Propositions 32.10 and 32.L [11].

Next, we introduce the nonlinear operator $S: V \to V^*$ naturally associated with the Kirchhoff-type nonlinearity appearing in the problem

$$\langle Su, v\rangle := \int_0^T M_A(t) \int_\Omega a(x,t,\nabla u)\nabla v\, dx\, dt.$$

As established in Lemma 4.2, the mapping S is bounded, demicontinuous, and of class (S_+). To establish coercivity of $L + S$, consider any $u \in D(L)$. Then,

$$\langle Lu + Su, u\rangle \geq \langle Su, u\rangle = \int_0^T M_A(t) \int_\Omega a(x,t,\nabla u) \times \nabla u\, dx\, dt.$$

Using the structure conditions on a and the monotonicity of M, we obtain

$$\langle Lu + Su, u\rangle \geq m_0 \int_0^T \int_\Omega [c_2|\nabla u(x,t)|^{p(x)} - k_2(x,t)]dxdt \geq c_2 m_0 \int_0^T \int_\Omega |\nabla u(x,t)|^{p(x)}\, dx\, dt - m_0\|k_2\|_{L^1(Q)}.$$

From the generalized Poincaré inequality, integrability of k_2 and the embedding properties of variable exponent Sobolev spaces, we conclude that there exists a constant $C > 0$ such that

$$\langle Lu + Su, u\rangle \geq C\|\nabla u\|^{\bar{p}}_{L^{p(\cdot)}(Q)} - m_0\|k_2\|_{L^1(Q)},$$

for some $\bar{p} > 1$, whence coercivity follows

$$\langle Lu + Su, u\rangle \to +\infty \quad \text{as } \|u\|_V \to \infty.$$

By standard arguments, this coercivity implies that, for every $h \in V^*$, there exists $R = R(h) > 0$ such that

$$\langle Lu + Su - h, u\rangle > 0$$

for all $u \in D(L)$ with $\|u\|_V = R$. Therefore, by applying Theorem 3.2 on the topological degree for $L + S$, we conclude that the equation

$$Lu + Su = h$$

admits at least one weak solution $u \in D(L)$. This function satisfies the weak formulation of (1.1), completing the proof. Q.E.D.

5. Conclusion

We have shown the existence of weak solutions to a class of Kirchhoff-type parabolic equations with variable exponents. By extending topological degree methods to this nonlocal, nonlinear setting, our result highlights the versatility of degree theory in treating complex evolution problems with nonstandard structure.

REFERENCES

[1] D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces*, Springer, 2013.

[2] L. Diening, et al., *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, 2011.

[3] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory,* Springer, 2000.

[4] J. Giacomoni, et al., "Quasilinear parabolic problems with $p(x)$-Laplacian: existence and stabilization," *Nonlinear Differ. Equ. Appl.,* vol. 23, 2016.

[5] P. Harjulehto, et al., "The Dirichlet energy integral and variable exponent Sobolev spaces," *Potential Anal.,* vol. 25, p. 205–222, 2006.

[6] M. Ait Hammou, "Quasilinear parabolic problem with $p(x)$-Laplacian operator by topological degree," *Kragujevac J. Math,* vol. 47, no. 4, p. 523–529, 2023.

[7] J. Berkovits, V. Mustonen, "Topological degree for perturbations of linear maximal monotone mappings and applications," *Rend. Mat. Appl.,* vol. 12, no. 3, p. 597–621, 1992.

[8] M. Bendahmane, P. Wittbold and A. Zimmermann, "Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1 -data," *J. Differential Equations,* vol. 249, p. 1483–1515, 2010.

[9] C. Yazough, "Existence solutions for a class of nonlinear parabolic equations with variable exponents and L^1 data," *Gulf Journal of Mathematics,* vol. 6, no. 4, pp. 56-71, 2018.

[10] D. Blanchard, F. Murat, "Renormalized solutions of nonlinear parabolic problems with L^1-data, Existence and Uniqueness," *Proc. Roy. Soc. Edinburgh Sect.,* no. A127, pp. 1137-1152., 1997.

[11] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Springer-Verlag, New York., 1990.

NGHIỆM YẾU CHO LỚP BÀI TOÁN PARABOLIC PHI ĐỊA PHƯƠNG KIỂU KIRCHHOFF BẰNG CÁCH SỬ DỤNG LÍ THUYẾT BẬC TOPO

Phạm Văn Dự

Trường Đại học Đồng Nai

Email: dupv@dnpu.edu.vn

(Ngày nhận bài: 13/6/2025, ngày nhận bài chỉnh sửa: 11/8/2025, ngày duyệt đăng: 24/10/2025)

TÓM TẮT

Chúng tôi nghiên cứu một lớp các phương trình parabolic phi tuyến, phi địa phương, có đặc trưng bởi một thành phần kiểu Kirchhoff và sự tăng trưởng theo số mũ biến. Các phương trình này xuất hiện tự nhiên trong mô hình hóa các quá trình vật lí phức tạp như chất lỏng điện môi và khuếch tán bất đẳng hướng, nơi sự tương tác giữa *điều kiện tăng trưởng phi chuẩn và các hiệu ứng phi cục bộ đóng vai trò quan trọng. Lấy cảm hứng từ đóng góp gần đây của M. Ait Hammou [Kragujevac J. Math, 47(4), 523–529 (2023)], người đã sử dụng lí thuyết bậc topo để giải quyết các bài toán parabolic quasilinear liên quan đến toán tử* $p(x)$ *−Laplacian, chúng tôi mở rộng khung phân tích này cho các toán tử kiểu Leray–Lions phụ thuộc gradient, kết hợp với các hệ số phi cục bộ kiểu Kirchhoff. Dựa vào nền tảng trên, chúng tôi sử dụng một phiên bản tinh chỉnh của lí thuyết bậc Berkovits–Mustonen để chứng minh sự tồn tại của nghiệm yếu trong các không gian Sobolev với số mũ biến. Hướng tiếp cận kết quả của chúng tôi là dựa vào việc xác minh các tính chất quan trọng của toán tử liên quan — chẳng hạn như tính liên tục, bị chặn và điều kiện* (S_+) *— để áp dụng lập luận điểm về lí thuyết Bậc Topo cho tổng các toán tử. Bài báo này giúp mở rộng phạm vi ứng dụng của các phương pháp Bậc Topo đến một lớp rộng hơn các phương trình parabolic phi tuyến, bao gồm nhiều mô hình với khuếch tán phi chuẩn và các hiệu ứng phi cục bộ.*

Từ khóa: *Các bài toán parabolic phi địa phương, toán tử* $p(x)$ *−Laplacian, bậc tôpô, số mũ biến*